Additional questions for Chapter 1 - solutions:

1. See Varian, p.20. Notice that Varian set a1=a2.

2. See Varian, p.20.

3. See Varian, p.20. Idea: use the TRS…

4.

Calculating the output elasticities for each input, we obtain

$$\mu_1(\mathbf{x}) = \alpha\left(1 + x_1^{-\alpha}x_2^{-\beta}\right)^{-1}x_1^{-\alpha}x_2^{-\beta},$$
$$\mu_2(\mathbf{x}) = \beta\left(1 + x_1^{-\alpha}x_2^{-\beta}\right)^{-1}x_1^{-\alpha}x_2^{-\beta},$$

each of which clearly varies with both scale and input proportions. Adding the two gives the following expression for the elasticity of scale:

$$\mu(\mathbf{x}) = (\alpha + \beta)\left(1 + x_1^{-\alpha}x_2^{-\beta}\right)^{-1}x_1^{-\alpha}x_2^{-\beta},$$

which also varies with $\mathbf{x}$.

Additional questions for Chapter 2 - solutions:

1.

Let the production function be the CES form,

$$y = \left(x_1^\rho + x_2^\rho\right)^{\beta/\rho}.$$

Suppose, therefore, that $\beta < 1$ and that $0 \neq \rho < 1$.

Form the Lagrangian for the profit-maximization problem in (3.6). By assuming an interior solution, the first-order conditions reduce to

$$-w_1 + p\beta\left(x_1^\rho + x_2^\rho\right)^{(\beta-\rho)/\rho} x_1^{\rho-1} = 0, \tag{E.1}$$

$$-w_2 + p\beta\left(x_1^\rho + x_2^\rho\right)^{(\beta-\rho)/\rho} x_2^{\rho-1} = 0, \tag{E.2}$$

$$\left(x_1^\rho + x_2^\rho\right)^{\beta/\rho} - y = 0. \tag{E.3}$$

Taking the ratio of (E.1) to (E.2) gives $x_1 = x_2(w_1/w_2)^{1/(\rho-1)}$. Substituting in (E.3) gives

$$x_i = y^{1/\beta}\left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)}\right)^{-1/\rho} w_i^{1/(\rho-1)}, \qquad i = 1, 2. \tag{E.4}$$

Substituting these into (E.1) and solving for y gives the supply function,

$$y = (p\beta)^{-\beta/(\beta-1)}\left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)}\right)^{\beta(\rho-1)/\rho(\beta-1)}. \tag{E.5}$$

From (E.4) and (E.5), we obtain the input demand functions,

$$x_i = w_i^{1/(\rho-1)}(p\beta)^{-1/(\beta-1)}\left(w_1^{\rho/(\rho-1)} + w_2^{\rho/(\rho-1)}\right)^{(\rho-\beta)/\rho(\beta-1)}, \qquad i = 1, 2. \tag{E.6}$$

To form the profit function, substitute from these last two equations into the objective function to obtain

$$\pi(p, \mathbf{w}) = p^{-1/(\beta-1)}\left(w_1^r + w_2^r\right)^{\beta/r(\beta-1)}\beta^{-\beta/(\beta-1)}(1-\beta), \tag{E.7}$$

where we've let $r \equiv \rho/(\rho-1)$.

Notice that if $\beta = 1$, the production function has constant returns and the profit function is undefined, as we concluded earlier. If $\beta > 1$, and the production function exhibits increasing returns, we could certainly form (E.7) as we have, but what would it give us? If you look closely, and check the second-order conditions, you'll find that (E.5) and (E.6) give a local profit *minimum*, not maximum. Maximum profits with increasing returns is similarly undefined. □

If B < 1,

Y(tx) = … = $t^{Bp/p}$ *(x_1^p = x_2^p). If B < 1 & we know 0<p <1, then BP < p, $t^{Bp/p}$ then < t, then DRS.

2.

Let's derive the short-run profit function for the constant-returns Cobb-Douglas technology. Supposing that x_2 is fixed at $\bar{x}_2$, our problem is to solve:

$$\max_{y, x_1} \; py - w_1 x_1 - \bar{w}_2 \bar{x}_2 \qquad \text{s.t.} \qquad x_1^{\alpha} \bar{x}_2^{1-\alpha} \geq y,$$

where $0 < \alpha < 1$. Assuming an interior solution, the constraint holds with equality, so we can substitute from the constraint for y in the objective function. The problem reduces to choosing the single variable x_1 to solve:

$$\max_{x_1} \; p x_1^{\alpha} \bar{x}_2^{1-\alpha} - w_1 x_1 - \bar{w}_2 \bar{x}_2. \tag{E.1}$$

The first-order condition on choice of x_1 requires that

$$\alpha p x_1^{\alpha-1} \bar{x}_2^{1-\alpha} - w_1 = 0.$$

Solving for x_1 gives

$$x_1 = p^{1/(1-\alpha)} w_1^{1/(\alpha-1)} \alpha^{1/(1-\alpha)} \bar{x}_2. \tag{E.2}$$

Substituting into (E.1) and simplifying gives the short-run profit function,

$$\pi(p, w_1, \bar{w}_2, \bar{x}_2) = p^{1/(1-\alpha)} w_1^{\alpha/(\alpha-1)} \alpha^{\alpha/(1-\alpha)} (1-\alpha) \bar{x}_2 - \bar{w}_2 \bar{x}_2. \tag{E.3}$$

Notice that because $\alpha < 1$, short-run profits are well-defined even though the production function exhibits (long-run) constant returns to scale.

By Hotelling's lemma, short-run supply can be found by differentiating (E.3) with respect to p:

$$\begin{aligned} y(p, w_1, \bar{w}_2, \bar{x}_2) &= \frac{\partial \pi(p, w_1, \bar{w}_2, \bar{x}_2)}{\partial p} \\ &= p^{\alpha/(1-\alpha)} w_1^{\alpha/(\alpha-1)} \alpha^{\alpha/(1-\alpha)} \bar{x}_2. \end{aligned}$$

3.

$$\max_{x,y} x^2 + y^2 + y - 1$$
$$\text{s.t.: } (x, y) \in \mathcal{D} = \{(x, y) \in \mathbb{R}^2 \mid g(x, y) = x^2 + y^2 \leq 1\}.$$

1. Step: Set up the Lagrangian as

$$\mathcal{L}(x, y, \lambda) = x^2 + y^2 + y - 1 - \lambda(x^2 + y^2 - 1).$$

2. Step: The critical points are the solutions (x, y, λ) to

$$2x - 2\lambda x = 0$$
$$2y + 1 - 2\lambda y = 0$$
$$\lambda \geq 0 \ (= 0 \text{ if } x^2 + y^2 < 1).$$

The first condition tells us that either $\lambda = 1$ or $x = 0$. If $\lambda = 1$, then the second condition yields $1 = 0$, a contradiction. Thus $x = 0$ and $y = \pm 1$ because in the first case we take λ to be nonzero. We conclude that the candidates for optimality in this case are $f(0, 1)$ and $f(0, -1)$.

Now we check the case where $x = 0$ and $\lambda = 0$ and therefore $x^2 + y^2 < 1 \Leftrightarrow -1 < y < 1$. Then the third condition implies $\lambda = 0$ and therefore the second condition says $y = -\frac{1}{2}$. Therefore $f(0, -\frac{1}{2})$ is our third candidate.

3. Step: Evaluate the candidates

$$f(0, 1) = 1$$
$$f(0, -1) = -1 \qquad (4.2.3)$$
$$f(0, -\tfrac{1}{2}) = -\tfrac{5}{4}.$$

Let us now check whether the two conditions we attached to the application of the Kuhn-Tucker Theorem hold.

Condition: (Existence) Since we maximise a continuous function $f(\cdot)$ over a closed and bounded set $\mathcal{D}$, we know that there is a solution to the problem. We conclude that $f(0, 1) = 1$ actually solves the problem.

Condition: (Constraint qualification) The constraint gradient is given by $\nabla g(x, y) = (2x, 2y)$ which, for all the possible candidates in (4.2.3), always has rank 1. Therefore our constraint qualification holds.$\triangleleft$

4.

This example is taken from (Sundaram 1996, 155). Let $g(x,y) = 1 - x^2 - y^2$. Consider the problem of maximising the objective $f(x,y) = x^2 - y$

$$\max_{x,y} x^2 - y$$
$$\text{s.t.: } (x,y) \in \mathcal{D} = \{(x,y) \in \mathbb{R}^2 \mid g(x,y) = 1 - x^2 - y^2 \geq 0\}.$$

1. Step: Set up the Lagrangian as

$$\mathcal{L}(x,y,\lambda) = x^2 - y + \lambda(1 - x^2 - y^2).$$

2. Step: The critical points are the solutions (x,y,λ) to

$$2x - 2\lambda x = 0$$
$$-1 - 2\lambda y = 0$$
$$\lambda \geq 0,\ (1 - x2 - y2) \geq 0,\ \lambda(1 - x2 - y2) = 0.$$

For the first of these to hold we must have either $x = 0$ or $\lambda = 1$. If $\lambda = 1$, the second equation gives $y = -\frac{1}{2}$ and $x^2 + y^2 = 1$ follows from the third. Therefore we get a set of these two points (x,y,λ) and a third for the case of $x = 0$ and $\lambda > 0$

$$M = \left\{(+\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1), (-\frac{\sqrt{3}}{2}, -\frac{1}{2}, 1), (0, -1, \frac{1}{2})\right\}$$

with $f(x,y)_{1,2} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$ for the first two and $f(x,y)_3 = 1$ for the third.

3. Step: We know that a global maximum for f on $\mathcal{D}$ must arise on one of the three points in M. Evidently the first two are equally good and better than the third. Hence we have two solutions to the given optimisation problem, namely the two points $(x,y) \in \left\{(+\frac{\sqrt{3}}{2}, -\frac{1}{2}), (-\frac{\sqrt{3}}{2}, -\frac{1}{2})\right\}$.

Again we check whether the two conditions we attached to the application of the KT Theorem hold.

Condition: (Existence) Our feasible set $\mathcal{D}$ is the closed unit disk—this set is compact. Since the objective is continuous (all polynomials are continuous), we know by the Weierstrass Theorem that a maximum exists. Thus the first condition above is met.

Condition: (Constraint qualification) The constraint qualification is also met: At a point (x,y) where the constraint binds (i.e., $(x,y) \in \mathcal{D}$), we have $x^2 + y^2 = 1$ and we know that either $x \neq 0$ or $y \neq 0$. Since $\nabla g(x,y) = (-2x, -2y)$ at all $(x,y) \in \mathcal{D}$ it follows that when g is effective, we must have $rank(\nabla g(x,y)) = 1$. Thus the constraint qualification holds for if the optimum occurs at a (x,y) where $g(x,y) = 0$. If the optimum occurs at a (x,y) where $g(x,y) > 0$, no constraints are binding, the set $\mathcal{D}$ is empty and the constraint qualification is met trivially.$\triangleleft$

– Handout –

–

This problem concerns a price-taking firm that produces two outputs, goods 2 and 3, using a single input, good 1. Its implicit production function (aka transformation function) is:

$$F : \Re^3 \to \Re : F(y_1, y_2, y_3) = 2(y_2)^2 + 3(y_3)^2 + y_1,$$

so that any vector (y_1, y_2, y_3) satisfying $y_1 \leq 0$ and $2(y_2)^2 + 3(y_3)^2 + y_1 \leq 0$ is technologically feasible. Moreover, its production set satisfies the free disposal assumption.

1 *Does the technology display nonincreasing returns to scale?*[1] *Show your work and explain.*

Nonincreasing returns to scale requires that for any $y \in Y$ and any $t \in [0, 1]$, then $ty \in Y$. The production set Y is only partially defined by $F(y) \leq 0 \Rightarrow y \in Y$, as there are other points in Y as a consequence of the free disposal assumption. However, we can limit ourselves to those vectors such that $F(y) \leq 0$. Therefore nonincreasing returns to scale is satisfied if whenever $F(y) \leq 0$ and $t \in [0, 1]$, we then also have $F(ty) \leq 0$.

Let $y \in \Re^3$ such that $F(y) \leq 0$ and let $t \in [0, 1]$. In particular, we then have $2(y_2)^2 + 3(y_3)^2 + y_1 \leq 0$. Then,

$$\begin{aligned} F(ty) &= 2(ty_2)^2 + 3(ty_3)^2 + ty_1 = t^2\left(2\left(y_2\right)^2 + 3\left(y_3\right)^2\right) + ty_1 \\ &= t\left(t\left(2\left(y_2\right)^2 + 3\left(y_3\right)^2\right) + y_1\right) \\ &\leq t\left(\left(2\left(y_2\right)^2 + 3\left(y_3\right)^2\right) + y_1\right) = t\left(2\left(y_2\right)^2 + 3\left(y_3\right)^2 + y_1\right). \end{aligned}$$

From $2(y_2)^2 + 3(y_3)^2 + y_1 \leq 0$ and $t \in [0, 1]$ it follows that $t\left(2\left(y_2\right)^2 + 3\left(y_3\right)^2 + y_1\right) \leq 0$ and therefore $F(ty) \leq 0$. This shows that the technology displays nonincreasing returns to scale.

2 *Write the PROFITMAX Problem.*

The $PROFITMAX[p]$ problem is

$$\begin{aligned} \max_{y_1, y_2, y_3 \in \Re^3} \quad & p \cdot y \\ s.t. \quad & y_1 \leq 0, \\ & 2{y_2}^2 + 3{y_3}^2 + y_1 \leq 0. \end{aligned}$$

[1] *Your answer must be based on the definition presented in class (and in the textbook); alternative definitions of this term might not be equivalent.*

3 *Obtain the supply function and profit function. (assuming price-taking in all input and output markets).*

Since ${y_2}^2 \geq 0$ and ${y_3}^2 \geq 0$, and hence $2{y_2}^2 + 3{y_3}^2 \geq 0$, it follows from $F(y) \leq 0$ that $y_1 \leq 0$. Therefore the $y_1 \leq 0$ constraint is redundant.

Let $\hat{y} \in Y$ such that $F(\hat{y}) < 0$, or $(\hat{y}_2)^2 + 3(\hat{y}_3)^2 + \hat{y}_1 < 0$. (The production vector $\hat{y}$ is feasible but is not on the transformation function frontier.) Then for some positive value Δy_1 we have $(\hat{y}_2)^2 + 3(\hat{y}_3)^2 + \hat{y}_1 + \Delta y_1 = 0$. The increment Δy_1 reduces the negative value of y_1 so that we are using less of this input. Then profit is increased by lowering this input expenditure by $p_1 \Delta y_1$. It follows that $\hat{y}$ cannot be profit maximizing. Therefore we can limit our attention to production vectors on the the transformation function frontier defined by $2{y_2}^2 + 3{y_3}^2 + y_1 = 0$.

With these two observations we know that we can instead work with this simpler equivalent profit maximization problem with only constraint, an equality constraint:

$$\begin{aligned} \max_{y_1, y_2, y_3 \in \Re^3} \quad & p_1 y_1 + p_2 y_2 + p_3 y_3 \\ s.t. \quad & 2{y_2}^2 + 3{y_3}^2 + y_1 = 0. \end{aligned}$$

The Lagrangian function is then

$$\mathcal{L}(y_1, y_2, y_3, \lambda) = p_1 y_1 + p_2 y_2 + p_3 y_3 - \lambda\left(2{y_2}^2 + 3{y_3}^2 + y_1\right)$$

with first order conditions,

$$\begin{aligned} (\mathrm{A}_1): \quad & p_1 - \lambda &= 0, \\ (\mathrm{A}_2): \quad & p_2 - 4\lambda y_2 &= 0, \\ (\mathrm{A}_3): \quad & p_3 - 6\lambda y_3 &= 0, \\ (\mathrm{B}): \quad & 2{y_2}^2 + 3{y_3}^2 + y_1 &= 0 \end{aligned}$$

From these, we can obtain the component supply functions in the following order,

$$\begin{aligned} \widetilde{y}_2(p_1, p_2, p_3) &= \frac{p_2}{4p_1}, \\ \widetilde{y}_3(p_1, p_2, p_3) &= \frac{p_3}{6p_1}, \\ \widetilde{y}_1(p_1, p_2, p_3) &= -\left(2\left(\frac{p_2}{4p_1}\right)^2 + 3\left(\frac{p_3}{6p_1}\right)^2\right) \\ &= -\left(\frac{1}{8}\left(\frac{p_2}{p_1}\right)^2 + \frac{1}{12}\left(\frac{p_3}{p_1}\right)^2\right) \\ &= \frac{-1}{4{p_1}^2}\left(\frac{{p_2}^2}{2} + \frac{{p_3}^2}{3}\right) \\ &= -\frac{3{p_2}^2 + 2{p_3}^2}{24{p_1}^2}, \end{aligned}$$

and the profit function,

$$\begin{aligned}
\Pi(p) &= p_1 \frac{-1}{4{p_1}^2}\left(\frac{{p_2}^2}{2} + \frac{{p_3}^2}{3}\right) + p_2 \frac{p_2}{4p_1} + p_3 \frac{p_3}{6p_1} \\
&= \frac{-1}{4p_1}\left(\frac{{p_2}^2}{2} + \frac{{p_3}^2}{3}\right) + \frac{1}{2p_1}\left(\frac{{p_2}^2}{2} + \frac{{p_3}^2}{3}\right) \\
&= \frac{1}{4p_1}\left(\frac{{p_2}^2}{2} + \frac{{p_3}^2}{3}\right) \\
&= \frac{3{p_2}^2 + 2{p_3}^2}{24p_1}.
\end{aligned}$$

4 *Verfy that Hotelling's Lemma applies with all three commodities, input and outputs.*

Hotelling's lemma states that

$$\frac{\partial \Pi(p)}{\partial p_\ell} = \widetilde{y}_\ell(p), \ \forall\, \ell = 1, \ldots, L.$$

In our case,

$$\begin{aligned}
\frac{\partial \Pi(p)}{\partial p_1} &= (-1)\frac{1}{4}\frac{1}{{p_1}^2}\left(\frac{{p_2}^2}{2} + \frac{{p_3}^2}{3}\right) = \widetilde{y}_1(p), \ \checkmark \\
\frac{\partial \Pi(p)}{\partial p_2} &= \frac{1}{4p_1}\left(2\frac{p_2}{2}\right) = \widetilde{y}_2(p), \ \checkmark \\
\frac{\partial \Pi(p)}{\partial p_3} &= \frac{1}{4p_1}\left(2\frac{p_3}{3}\right) = \widetilde{y}_3(p), \ \checkmark.
\end{aligned}$$

5 *Show that the simple law of supply applies to both outputs and that the simple law of demand applies the single factor of production.*

Supply:

$$\begin{aligned}
\frac{\partial \widetilde{y}_2}{\partial p_2}(p_1, p_2, p_3) &= \frac{1}{4p_1} > 0, \\
\frac{\partial \widetilde{y}_3}{\partial p_3}(p_1, p_2, p_3) &= \frac{1}{6p_1} > 0.
\end{aligned}$$

Thus both outputs increase with their own price, satisfying the law of supply.

Factor Demand:

$$\frac{\partial \widetilde{y}_1}{\partial p_1}(p_1, p_2, p_3) = (-2)\frac{-1}{4{p_1}^3}\left(\frac{{p_2}^2}{2} + \frac{{p_3}^2}{3}\right) = \frac{1}{2{p_1}^3}\left(\frac{{p_2}^2}{2} + \frac{{p_3}^2}{3}\right) > 0.$$

This equation shows that y_1 increases in value on the real line with its own price. However, since $y_1 \leq 0$, this means that y_1 is moving towards zero from the left. Therefore the absolute value of y_1 is getting smaller so that in fact there is a smaller demand for the input with a higher input price. Thus, the the law of demand is satisfied.

Louisiana State University Economics 7720
Fall 2014

– Answer Key –

General CES Production.

For this problem set you are going to work with a more general version of CES production than presented in class. Let the production function $f : \Re^2_+ \to \Re$ *be defined by:*

$$f(x) \,=\, \left(a_1\,(x_1)^\rho + a_2\,(x_2)^\rho\right)^{\alpha/\rho},$$

where $a_1 > 0$, $a_2 > 0$, $\alpha > 0$ *and* $\rho \in (-\infty, 0) \bigcup (0, 1)$.

1. *Carefully state this production technology as a transformation function. Include all appropriate restrictions on independent variable values.*

   We have two inputs and a single output so that $Y \subseteq \Re^3$. For each netput vector $y \in \Re^3$, let y_1 and y_2 be the two respective inputs and let y_3 be the output.[1] Recall that the convention with transformation functions requires all inputs be stated as negative values, so that $y_1 = -x_1$ and $y_2 = -x_2$. Therefore, in creating the transformation function from the production function we need to substitute for x_1 and x_2 with $x_1 = -y_1$ and $x_2 = -y_2$. Then with technical efficiency we have

$$\begin{aligned} y_3 \,&=\, \left(a_1\,(-y_1)^\rho + a_2\,(-y_2)^\rho\right)^{\alpha/\rho}, \\ y_3 - \left(a_1\,(-y_1)^\rho + a_2\,(-y_2)^\rho\right)^{\alpha/\rho} \,&=\, 0. \qquad (1) \end{aligned}$$

   Note that with technical inefficiency,[2] the left hand part of equation (1) is less than zero. Also, with combinations of inputs and output that are not feasible this same is expression is greater than zero. Thus the transformation function is $F : \Re^2_- \times \Re_+ \to \Re$,

$$F(y_1, y_2, y_3) \,=\, y_3 - \left(a_1\,(-y_1)^\rho + a_2\,(-y_2)^\rho\right)^{\alpha/\rho}.$$

2. *Specify the general input requirement set as a function of output, y.*

   The input requirement set specifies for any given output level, the feasible combinations of inputs. A combination of inputs is infeasible if the output level cannot be obtained, $f(x) < y$. Thus with feasibility, we may have either technical efficiency with $f(x) = y$ or inefficiency with $f(x) > y$. Therefore the input requirement set is

$$V(y) \,=\, \left\{(x_1, x_2) \in \Re^2_+ \mid \left(a_1\,(x_1)^\rho + a_2\,(x_2)^\rho\right)^{\alpha/\rho} \geq y \right\}.$$

[1] Alternatively, you could have let y_1 be the output, and y_2 and y_3 be the inputs.
[2] That is in the context of the production function, $f(x) < y$.

3. *Carefully show that this technology is monotone.*

In class I presented four different equivalent characterizations of production monotonicity, including two based on the input requirement set and one based on the transformation function. I will work with the transformation function. In that context, monotonicity requires that whenever $F(y) \leq 0$ (the netput vector is feasible) and $y' \leq y$, then $F(y') \leq 0$ (with both $y, y' \in \Re^N$). Working directly with this definition, let $y \in \Re^3$ with $F(y) \leq 0$ so that $y_3 - (a_1(-y_1)^\rho + a_2(-y_2)^\rho)^{\alpha/\rho} \leq 0$. Also let $y' \leq y$ for $y' \in \Re^3$. Then $-y_1' \geq -y_1$, $-y_2' \geq -y_2$ and $y_3' \leq y_3$.

We now need to work separately with the two ranges for values of ρ. Starting with $\rho \in (0,1)$, we have $(-y_1')^\rho \geq (-y_1)^\rho$, $(-y_2')^\rho \geq (-y_2)^\rho$, and hence

$$\begin{aligned} \left(a_1(-y_1')^\rho + a_2(-y_2')^\rho\right)^{\alpha/\rho} &\geq \left(a_1(-y_1)^\rho + a_2(-y_2)^\rho\right)^{\alpha/\rho}, \\ y_3' - \left(a_1(-y_1')^\rho + a_2(-y_2')^\rho\right)^{\alpha/\rho} &\leq y_3 - \left(a_1(-y_1)^\rho + a_2(-y_2)^\rho\right)^{\alpha/\rho}. \end{aligned}$$

Thus $F(y') \leq F(y)$ and hence $F(y') \leq 0$.

Because negative powers are monotonically decreasing,[3] the range $\rho \in (-\infty, 0)$ is a little bit more tricky. First we have $-y_k' \geq -y_k \Leftrightarrow (-y_k')^\rho \leq (-y_k)^\rho$ for $k = 1, 2$, and hence $a_1(-y_1')^\rho + a_2(-y_2')^\rho \leq a_1(-y_1)^\rho + a_2(-y_2)^\rho$. Then since also $\alpha/\rho < 0$,

$$\left(a_1(-y_1')^\rho + a_2(-y_2')^\rho\right)^{\alpha/\rho} \geq \left(a_1(-y_1)^\rho + a_2(-y_2)^\rho\right)^{\alpha/\rho}.$$

The rest follows as before.

When you first work with a new concept such as monotone production, it is best to work directly with the definition to develop familiarity, as we just did. However there are sometimes other approaches. In this case with $F(y)$ differentiable, the three first-order partial derivatives are useful. Beginning with y_1,

$$\begin{aligned} \frac{\partial F}{\partial y_1}(y) &= -(\alpha/\rho)\left(a_1(-y_1)^\rho + a_2(-y_2)^\rho\right)^{(\alpha/\rho)-1} \rho a_1(-y_1)^{\rho-1}(-1) \\ &= \alpha a_1(-y_1)^{\rho-1}\left(a_1(-y_1)^\rho + a_2(-y_2)^\rho\right)^{(\alpha/\rho)-1}. \end{aligned}$$

Since each of these factors is positive (with y_1 positive), $\partial F/\partial y_1(y) > 0$. Similarly $\partial F/\partial y_2(y) > 0$, and $\partial F/\partial y_3(y) > 0$ is trivial. It then follows that whenever $y' \leq y$, we also have $F(y') \leq F(y)$ and hence $F(y') \leq 0$.

[3]For $x, y \in \Re_+$ and $a < 0$, $x \leq y \Leftrightarrow x^a \geq y^a$.

4. *Provide an exact characterization of all parameter value combinations such that the consequent production technology has decreasing returns to scale. Show your work (i.e., justify your characterization).*

   One way of providing an exact characterization is with set notation. For example with the subset of real numbers $\widehat{R} = \rho \in (-\infty, 0) \bigcup (0, 1)$, you could define this set with the form,

$$\left\{(a_1, a_2, \alpha, \rho) \in \Re^3_{++} \times \widehat{R} \,|\, \ldots \ldots \right\}.$$

   Again working directly with our definition for transformation functions, decreasing returns-to-scale requires that whenever $F(y) = 0$ for $y \in \Re^3$ $(y \neq 0)$ and $t > 1$, we also have $F(ty) > 0$. In this case,

$$\begin{aligned} F(ty) &= ty_3 - \left(a_1\left(-ty_1\right)^\rho + a_2\left(-ty_2\right)^\rho\right)^{\alpha/\rho} \\ &= ty_3 - \left(a_1 t^\rho\left(-y_1\right)^\rho + a_2 t^\rho\left(-y_2\right)^\rho\right)^{\alpha/\rho} \\ &= ty_3 - \left(t^\rho\left(a_1\left(-y_1\right)^\rho + a_2\left(-y_2\right)^\rho\right)\right)^{\alpha/\rho} \\ &= ty_3 - \left(t^\rho\right)^{\alpha/\rho}\left(a_1\left(-y_1\right)^\rho + a_2\left(-y_2\right)^\rho\right)^{\alpha/\rho} \\ &= ty_3 - t^\alpha\left(a_1\left(-y_1\right)^\rho + a_2\left(-y_2\right)^\rho\right)^{\alpha/\rho}. \end{aligned}$$

   From $F(y) = 0$ we have $y_3 = \left(a_1\left(-y_1\right)^\rho + a_2\left(-y_2\right)^\rho\right)^{\alpha/\rho} > 0$, so that

$$F(ty) = ty_3 - t^\alpha y_3 = \left(t - t^\alpha\right) y_3.$$

   Then $F(ty) > 0 \Leftrightarrow t - t^\alpha > 0 \Leftrightarrow t > t^\alpha \Leftrightarrow \alpha < 1$. Thus whether or not this technology exhibits decreasing returns-to-scale depends only on the value of α and not at all on the other parameter values. With the suggested set notation this becomes

$$\left\{(a_1, a_2, \alpha, \rho) \in \Re^3_{++} \times \widehat{R} \,|\, \alpha \in (0, 1) \right\}.$$

5. *Is this production technology homothetic (perhaps for only some parameter value combinations)? Fully justify your answer.*

   Homothetic technology is defined in terms of the production function. It requires that $f = T \circ g$ where $g : \Re^2_+ \to \Re$ is homogeneous of degree one (HOD-1) and $T : \Re \to \Re$ is an increasing monotonic function. For this we need to find out if f is homogeneous (and of what degree),

$$\begin{aligned} f(tx) &= \left(a_1\left(tx_1\right)^\rho + a_2\left(tx_2\right)^\rho\right)^{\alpha/\rho} \\ &= \left(a_1 t^\rho\left(x_1\right)^\rho + a_2 t^\rho\left(x_2\right)^\rho\right)^{\alpha/\rho} \\ &= \left(t^\rho\left(a_1\left(x_1\right)^\rho + a_2\left(x_2\right)^\rho\right)\right)^{\alpha/\rho} \\ &= \left(t^\rho\right)^{\alpha/\rho}\left(a_1\left(x_1\right)^\rho + a_2\left(x_2\right)^\rho\right)^{\alpha/\rho} \\ &= t^\alpha\left(a_1\left(x_1\right)^\rho + a_2\left(x_2\right)^\rho\right)^{\alpha/\rho} \\ &= t^\alpha f(x). \end{aligned}$$

   Thus f is HOD-α. It follows that with $T(h) = h^\alpha$ and $g(x) = \left(a_1\left(x_1\right)^\rho + a_2\left(x_2\right)^\rho\right)^{1/\rho}$ HOD-1, we have $f(x) = T(g(x))$, so that f is homothetic.

6. *Provide the Technical Rate of Substitution of input 1 in terms of input 2 for this technology as a function in the form $TRS_{1,2} = \ldots$. Show your work, starting with the production function provided above.*

   I shall work with the general equation that we found,

$$TRS_{1,2}(x) = \frac{\dfrac{\partial f}{\partial x_1}(x)}{\dfrac{\partial f}{\partial x_2}(x)}.$$

   Beginning with x_1,

$$\begin{aligned}\frac{\partial f}{\partial x_1}(x) &= (\alpha/\rho)\left(a_1(x_1)^\rho + a_2(x_2)^\rho\right)^{(\alpha/\rho)-1}\rho a_1(x_1)^{\rho-1} \\ &= \alpha a_1(x_1)^{\rho-1}\left(a_1(x_1)^\rho + a_2(x_2)^\rho\right)^{(\alpha/\rho)-1}.\end{aligned}$$

   From the symmetry of the production function we also have

$$\frac{\partial f}{\partial x_2}(x) = \alpha a_2(x_2)^{\rho-1}\left(a_1(x_1)^\rho + a_2(x_2)^\rho\right)^{(\alpha/\rho)-1},$$

   so that

$$\begin{aligned}TRS_{1,2}(x) &= \frac{\dfrac{\partial f}{\partial x_1}(x)}{\dfrac{\partial f}{\partial x_2}(x)} \\ &= \frac{\alpha a_1(x_1)^{\rho-1}\left(a_1(x_1)^\rho + a_2(x_2)^\rho\right)^{(\alpha/\rho)-1}}{\alpha a_2(x_2)^{\rho-1}\left(a_1(x_1)^\rho + a_2(x_2)^\rho\right)^{(\alpha/\rho)-1}} \\ &= \frac{a_1(x_1)^{\rho-1}}{a_2(x_2)^{\rho-1}}.\end{aligned}$$

   Thus,

$$TRS_{1,2}(x) = \frac{a_1}{a_2}\left(\frac{x_1}{x_2}\right)^{\rho-1}.$$

   Note that this is the same as what we found in class with the regular CES production function (without α). This is discussed in the context of problem 9 below.

7. *Provide the gradient for this production function. Show your work.*

   The gradient of the production function is simply the vector of first partials which we have already found above,

$$\begin{aligned}\nabla f(x) &= \begin{pmatrix}\dfrac{\partial f}{\partial x_1}(x) \\ \\ \dfrac{\partial f}{\partial x_2}(x)\end{pmatrix} \\ &= \begin{pmatrix}\alpha a_1(x_1)^{\rho-1}\left(a_1(x_1)^\rho + a_2(x_2)^\rho\right)^{(\alpha/\rho)-1} \\ \\ \alpha a_2(x_2)^{\rho-1}\left(a_1(x_1)^\rho + a_2(x_2)^\rho\right)^{(\alpha/\rho)-1}\end{pmatrix}.\end{aligned}$$

Envelope theorem basic example:

1) Let f(x;a) = -x^2 + 2ax + 4a^2 be a function in one variable x that depends on a parameter a. Use the Envelope theorems to show that *df*/da* = 10a

In economic optimization problems, the objective functions that we try to maximize/minimize often depend on parameters, like prices. We want to find out how the optimal value is affected by changes in the parameters.

Example

Let $f(x;a) = -x^2 + 2ax + 4a^2$ be a function in one variable x that depends on a parameter a. For a given value of a, the stationary points of f is given by

$$\frac{\partial f}{\partial x} = -2x + 2a = 0 \quad \Leftrightarrow \quad x = a$$

and this is a (local and global) maximum point since $f(x;a)$ is concave considered as a function in x. We write $x^*(a) = a$ for the maximum point. The optimal value function $f^*(a) = f(x^*(a);a) = -a^2 + 2a^2 + 4a^2 = 5a^2$ gives the corresponding maximum value.

Envelope theorems

Envelope theorems: An example

Example (Continued)

The derivative of the value function is given by

$$\frac{\partial f^*}{\partial a} = \frac{\partial}{\partial a} f(x^*(a);a) = \frac{\partial}{\partial a}\left(5a^2\right) = 10a$$

On the other hand, we see that $f(x;a) = -x^2 + 2ax + 4a^2$ gives

$$\frac{\partial f}{\partial a} = 2x + 8a \quad \Rightarrow \quad \left(\frac{\partial f}{\partial a}\right)_{x=x^*(a)} = 2a + 8a = 10a$$

since $x^*(a) = a$.

Part 2 Cost minimization.

The technology of another firm can be represented by the production function $f : \Re^2_+ \to \Re$,

$$f(x_1, x_2) = \left(2\sqrt{x_1} + 3\sqrt{x_2}\right)^{3/2}.$$

2.1 (2pt.) *If you can, please name this type of technology.*

Recall that with your other problem set, you worked with a general production function $f : \Re^2_+ \to \Re$ defined by:

$$f(x) = \left(a_1 (x_1)^\rho + a_2 (x_2)^\rho\right)^{\alpha/\rho},$$

where $a_1 > 0$, $a_2 > 0$, $\alpha > 0$ and $\rho \in (-\infty, 0) \bigcup (0, 1)$. The production function for this part of the exam is clearly an example of that general production function, which was a variation on $\boxed{\text{CES}}$.

2.2 (4pt.) *Formally state the firm's cost minimization problem with all appropriate constraints in their appropriate form.*

The cost minimization problem can be stated either with the nonnegativity constraints implied with the min notation or explicitly listed:

$$\begin{aligned} \min_{(x_1,x_2)\in\Re^2_+} \quad & w_1x_1 + w_1x_2 \\ s.t. \quad & \left(2\sqrt{x_1} + 3\sqrt{x_2}\right)^{3/2} \geq y. \end{aligned}$$

$$\begin{aligned} \min_{(x_1,x_2)\in\Re^2} \quad & w_1x_1 + w_1x_2 \\ s.t. \quad & \left(2\sqrt{x_1} + 3\sqrt{x_2}\right)^{3/2} \geq y, \\ & x_1 \geq 0, \\ & x_2 \geq 0. \end{aligned}$$

2.3 (12pt.) *Please find the firms's two component conditional factor demand functions. These need to be explicitly written as functions. You need to be clear about how you deal with any inequality constraints.*

This time we have three inequality constraints, two nonnegativity constraints and production feasibility constraint. As with the nonpositivity constraint in part 1, I will defer these nonnegativity constraints until after the Lagrangian analysis. Suppose that $\left(2\sqrt{x_1} + 3\sqrt{x_2}\right)^{3/2} > y$. Then we can lower the x_1 or x_2 values at least some and still satisfy the constraint. Moreover, reduction in either value, by say Δx_k, will reduce the cost of inputs by $p_k \Delta x_k$. We have shown that $\left(2\sqrt{x_1} + 3\sqrt{x_2}\right)^{3/2} > y$ is not cost minimizing, and can assume the equality constraint $\left(2\sqrt{x_1} + 3\sqrt{x_2}\right)^{3/2} = y$.

Therefore we will work with the simpler optimization problem,

$$\begin{aligned} \min_{(x_1,x_2)\in\Re^2} \quad & w_1x_1 + w_1x_2 \\ s.t. \quad & \left(2\sqrt{x_1} + 3\sqrt{x_2}\right)^{3/2} = y. \end{aligned}$$

with Lagrangian,

$$\mathcal{L}(x,\lambda) = w_1x_1 + w_2x_2 - \lambda\left(\left(2\sqrt{x_1}+3\sqrt{x_2}\right)^{3/2} - y\right),$$

and first order conditions,

$$\begin{aligned}
(\mathrm{A}_1): \quad & w_1 - \lambda\frac{3}{2}\left(2\sqrt{x_1}+3\sqrt{x_2}\right)^{1/2}(2)\frac{1}{2}\frac{1}{\sqrt{x_1}} &= 0,\\
(\mathrm{A}_2): \quad & w_2 - \lambda\frac{3}{2}\left(2\sqrt{x_1}+3\sqrt{x_2}\right)^{1/2}(3)\frac{1}{2}\frac{1}{\sqrt{x_2}} &= 0,\\
(\mathrm{B}): \quad & \left(2\sqrt{x_1}+3\sqrt{x_2}\right)^{3/2} &= y.
\end{aligned}$$

From (A_1) and (A_2) we have

$$\frac{w_1}{w_2} = \frac{2}{3}\frac{\sqrt{x_2}}{\sqrt{x_1}}, \tag{1}$$

yielding the (non-binding) technical rate of substitution,

$$TRS_{1,2}(x_1,x_2) = \frac{2}{3}\sqrt{\frac{x_2}{x_1}}.$$

Going back to equation (1), at this point we would typically solve for either x_1 or x_2, and then substitute into (B). However, in this case a quick look at (B) suggests that it will be quicker to solve for either $\sqrt{x_1}$ or $\sqrt{x_2}$. Since I want one of these by itself on the right, I will solve for $\sqrt{x_1}$,

$$\sqrt{x_1} = \frac{2}{3}\frac{w_2}{w_1}\sqrt{x_2}, \tag{2}$$

Substituting this into (B) yields,

$$\begin{aligned}
\left(2\left(\frac{2}{3}\frac{w_2}{w_1}\sqrt{x_2}\right)+3\sqrt{x_2}\right)^{3/2} &= y\\
\sqrt{x_2}\left(\frac{4}{3}\frac{w_2}{w_1}+3\right) &= y^{2/3}\\
\sqrt{x_2}\,\frac{4w_2+9w_1}{3w_1} &= y^{2/3}\\
\sqrt{x_2} &= \frac{3w_1}{4w_2+9w_1}\,y^{2/3} \qquad (3)\\
x_2 &= \left(\frac{3w_1}{4w_2+9w_1}\right)^2 y^{4/3}. \qquad (4)
\end{aligned}$$

Substituting equation (2) into (3) gives us

$$\begin{aligned}
\sqrt{x_1} &= \frac{2}{3}\frac{w_2}{w_1}\frac{3w_1}{4w_2+9w_1}\,y^{2/3}\\
\sqrt{x_1} &= \frac{2w_2}{4w_2+9w_1}\,y^{2/3}\\
x_1 &= \left(\frac{2w_2}{4w_2+9w_1}\right)^2 y^{4/3} \qquad (5)
\end{aligned}$$

We have obtained equations (4) and (5) without imposing the nonnegativity constraints on both inputs. However both expressions are nonnegative for all values of the economic independent variables, $p_1 > 0$, $p_2 > 0$ and $y \geq 0$. Thus the nonnegativity constraints are always satisfied with this pair of expressions and we can use them as the requested component conditional factor demand functions,

$$\boxed{\widetilde{x}_1(w_1, w_2, y) = \left(\frac{2w_2}{9w_1 + 4w_2}\right)^2 y^{4/3} \quad \text{and} \quad \widetilde{x}_2(w_1, w_2, y) = \left(\frac{3w_1}{9w_1 + 4w_2}\right)^2 y^{4/3}}.$$

2.4 (6pt.) *Provide the cost function for this firm clearly stated as a function.*

The cost of production is the firm's expenditure on the two inputs,

$$\begin{aligned}
c(w_1, w_2, y) &= w_1 \widehat{x}_1(w_1, w_2, y) + w_2 \widehat{x}_2(w_1, w_2, y) \\
&= w_1 \left(\frac{2w_2}{9w_1 + 4w_2}\right)^2 y^{4/3} + w_2 \left(\frac{3w_1}{9w_1 + 4w_2}\right)^2 y^{4/3} \\
&= \left(w_1 (2w_2)^2 + w_2 (3w_1)^2\right) \left(\frac{1}{9w_1 + 4w_2}\right)^2 y^{4/3} \\
&= w_1 w_2 (4w_2 + 9w_1) \left(\frac{1}{9w_1 + 4w_2}\right)^2 y^{4/3} \\
&= \frac{w_1 w_2}{9w_1 + 4w_2} y^{4/3}.
\end{aligned}$$

Summarizing,

$$\boxed{c(w_1, w_2, y) = \frac{w_1 w_2}{9w_1 + 4w_2} y^{4/3}}.$$

2.5 (8pt.) *Suppose that this firm's supply function is* $\widehat{y}(p, w_1, w_2) = \left(\frac{3}{4} \frac{p(9w_1 + 4w_2)}{w_1 w_2}\right)^3$. *Provide the (regular) factor demand function for input one* <u>*and*</u> *the firm's profit function, both clearly stated as functions.*

For this problem full credit was given for correctly fully setting up each of these functions. I did not take off any points because of simplification.

A (regular) factor demand function can be found by substituting the supply function into the corresponding conditional factor demand function,

$$\begin{aligned}
\widehat{x}_1(p, w_1, w_2) &= \widetilde{x}_1(w_1, w_2, \widehat{y}(p, w_1, w_2)) \\
&= \left(\frac{2w_2}{9w_1 + 4w_2}\right)^2 (\widehat{y}(p, w_1, w_2))^{4/3} \\
&= \left(\frac{2w_2}{9w_1 + 4w_2}\right)^2 \left(\left(\frac{3}{4} \frac{p(9w_1 + 4w_2)}{w_1 w_2}\right)^3\right)^{4/3} \\
&= \left(\frac{2w_2}{9w_1 + 4w_2}\right)^2 \left(\frac{3}{4} \frac{p(9w_1 + 4w_2)}{w_1 w_2}\right)^4 \\
&= (2)^2 \left(\frac{3}{4}\right)^4 (p)^4 \frac{(w_2)^2}{(w_1 w_2)^4} (9w_1 + 4w_2)^2 \\
&= \frac{81}{64} \frac{(p)^4}{(w_1)^4 (w_2)^2} (9w_1 + 4w_2)^2.
\end{aligned}$$

Profit is revenue less cost, and revenue is the price of output times the quantity supplied,

$$
\begin{aligned}
\widehat{\pi}\,(p, w_1, w_2) &= p\,\widehat{y}(p, w_1, w_2) \;-\; c\,(w_1, w_2, \widehat{y}(p, w_1, w_2)) \\
&= p\,\widehat{y}(p, w_1, w_2) \;-\; \frac{w_1 w_2}{9w_1 + 4w_2}\;(\widehat{y}(p, w_1, w_2))^{4/3} \\
&= p\,\left(\frac{3}{4}\frac{p(9w_1 + 4w_2)}{w_1 w_2}\right)^3 - \frac{w_1 w_2}{9w_1 + 4w_2}\left(\left(\frac{3}{4}\frac{p(9w_1 + 4w_2)}{w_1 w_2}\right)^3\right)^{4/3} \\
&= \left(\frac{3}{4}\right)^3 (p)^4\,\frac{(9w_1 + 4w_2)^3}{(w_1 w_2)^3} - \frac{w_1 w_2}{9w_1 + 4w_2}\left(\frac{3}{4}\right)^4 (p)^4\,\frac{(9w_1 + 4w_2)^4}{(w_1 w_2)^4} \\
&= \left(\frac{3}{4}\right)^3 (p)^4\,\frac{(9w_1 + 4w_2)^3}{(w_1 w_2)^3} - \left(\frac{3}{4}\right)^4 (p)^4\,\frac{(9w_1 + 4w_2)^3}{(w_1 w_2)^3} \\
&= \left(\frac{3}{4}\right)^3 \left(1 - \frac{3}{4}\right) (p)^3\,\frac{(9w_1 + 4w_2)^3}{(w_1 w_2)^3} \\
&= \left(\frac{3}{4}\right)^3 \left(\frac{1}{4}\right) (p)^3\,\frac{(9w_1 + 4w_2)^3}{(w_1 w_2)^3} \\
&= \frac{27}{256}\,(p)^3\,\frac{(9w_1 + 4w_2)^3}{(w_1 w_2)^3}
\end{aligned}
$$

Additional Questions for chapter 5 - solutions

1-

Calculate the cost function and conditional input demands for the Leontief production function

$$y = \min\{\alpha x_1, \beta x_2\} \text{ for } \alpha > 0 \text{ and } \beta > 0.$$

Answer:

Because the production is a min-function, set the inside terms equal to find the optimal relationship between x_1 and x_2. In other words, $\alpha x_1 = \beta x_2$. For a given level of output y, we must have $y = \alpha x_1 = \beta x_2$. Rearrange this expression to derive the conditional input demands:

$$x_1(\mathbf{w}, y) = \frac{y}{\alpha} \qquad x_2(\mathbf{w}, y) = \frac{y}{\beta}.$$

The cost function is obtained by substituting the two conditional demands into the definition of cost:

$$c(\mathbf{w}, y) = w_1 x_1(\mathbf{w}, y) + w_2 x_2(\mathbf{w}, y) = \frac{w_1 y}{\alpha} + \frac{w_2 y}{\beta}.$$

Same idea for 3 inputs

2-

Calculate the cost function and the conditional input demands for the linear production function $y = \sum_{i=1}^{n} \alpha_i x_i$.

Answer Because the production function is linear, the inputs can be substituted for another. The most efficient input (i.e. input with the greatest marginal product/ price) will be used and the other inputs will not be used.

$$x_i(\mathbf{w}, y) = \begin{cases} \frac{y}{\alpha_i} & \text{if } \frac{\alpha_i}{w_i} > \frac{\alpha_j}{w_j} \forall j \neq i,\ j \in \{1, \ldots, n\} \\ 0 & \text{if } \frac{\alpha_i}{w_i} < \frac{\alpha_j}{w_j} \text{ for at least one } j \neq i,\ j \in \{1, \ldots, n\}. \end{cases}$$

The cost function is then $c(\mathbf{w}, y) = \frac{w_i y}{\alpha_i}$, where i is the input where

$$\frac{\alpha_i}{w_i} > \frac{\alpha_j}{w_j}\ \forall j \neq i,\ j \in 1, \ldots, n.$$